

UNIVERSITY OF WATERLOO
FACULTY OF ENGINEERING
Department of Electrical & Computer Engineering

NE 112 *Linear algebra for nanotechnology engineering*

7.1 The vector representation of a line

Douglas Wilhelm Harder, LEL, M.Math.
dwharder@uwaterloo.ca
dwharder@gmail.com

CC BY NC SA

1

The vector representation of a line

Introduction

- In this topic, we will
 - Describe the vector representation of a line
 - Derive an ideal representation
 - Determine the point on the line closest to a given point $\mathbf{w}$ using this ideal representation
 - Find a formula for the minimum distance from $\mathbf{w}$ to that line

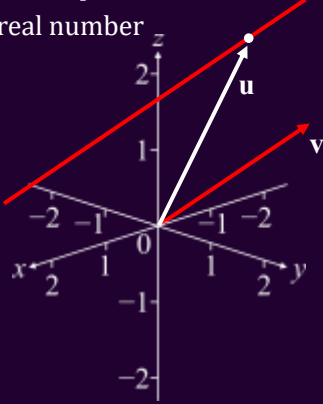
2

2

The vector representation of a line

Vector representation of a line

- In $\mathbf{R}^3$, a vector representation of a line is $\mathbf{u} + \alpha \mathbf{v}$ where
 - $\mathbf{u}$ is a point on the line
 - $\mathbf{v}$ is a non-zero vector parallel to the line
 - α takes on any real number



3

3

The vector representation of a line

Vector representation of a line

- Any line has infinitely many representations
 - For example, both of these describe the same line:

$$\begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix} + \alpha \begin{pmatrix} 2 \\ 1 \\ 8 \end{pmatrix} \quad \begin{pmatrix} 46.4 \\ 23.7 \\ 163.6 \end{pmatrix} + \alpha \begin{pmatrix} -75.8 \\ -37.9 \\ -303.2 \end{pmatrix}$$
 - Question: is any one representation better than others?

4

4

The vector representation of a line

Ideal vector representation

- First, recall that any point in 3-space will be associated with some form of units
 - Thus, if our measure is meters, then if $\|\mathbf{v}\|_2 = 1$, it follows that $\mathbf{u} + \alpha \mathbf{v}$ is $|\alpha|$ m from $\mathbf{u}$
- Also, the point $\mathbf{u}$ may be chosen to be any point on the line
 - Given any line, there is exactly one point that:
 - Has the smallest norm
 - That is, is closest to the origin
 - Is perpendicular to $\mathbf{v}$

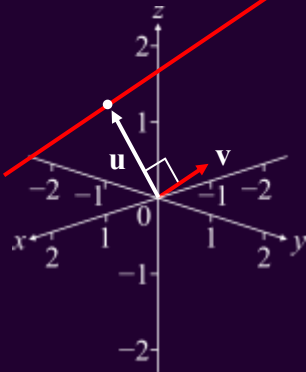
5

5

The vector representation of a line

Ideal vector representation

- Thus, we would prefer to represent a line with
 - The vector $\mathbf{v}$ being normalized
 - The vector $\mathbf{u}$ is perpendicular to the line



6

6

The vector representation of a line

Ideal vector representation

- Thus, given $\mathbf{u} + \alpha \mathbf{v}$, we can
 - First normalize $\mathbf{v}$, so the expression of our line is now

$$\mathbf{u} + \alpha \hat{\mathbf{v}}$$
 - Second, find the component of $\mathbf{u}$ perpendicular to $\hat{\mathbf{v}}$

$$\mathbf{u}_{\perp} \leftarrow \mathbf{u} - \langle \hat{\mathbf{v}}, \mathbf{u} \rangle \hat{\mathbf{v}}$$

7 

7

The vector representation of a line

Example

- For example, consider the line described by:

$$\begin{pmatrix} 18 \\ -23 \\ -2 \end{pmatrix} + \alpha \begin{pmatrix} 12 \\ -15 \\ 16 \end{pmatrix}$$
 - First, normalize the vector $\mathbf{v}$:

$$\begin{pmatrix} 18 \\ -23 \\ -2 \end{pmatrix} + \alpha \begin{pmatrix} 0.48 \\ -0.6 \\ 0.64 \end{pmatrix}$$
 - Next, find the component of $\mathbf{u}$ perpendicular to $\hat{\mathbf{v}}$

$$\begin{pmatrix} 7.8432 \\ -10.304 \\ -15.5424 \end{pmatrix} + \alpha \begin{pmatrix} 0.48 \\ -0.6 \\ 0.64 \end{pmatrix}$$

8 

8

The vector representation of a line

Point on the line closest to a given point

- Now, given we have $\mathbf{u}_\perp + \alpha \hat{\mathbf{v}}$,
what is the point on this line closest to a vector $\mathbf{w}$?
 - Because $\mathbf{u}_\perp$ is already perpendicular to $\hat{\mathbf{v}}$,
all we need to is project $\mathbf{w}$ onto $\hat{\mathbf{v}}$:

$$\mathbf{u}_\perp + \langle \hat{\mathbf{v}}, \mathbf{w} \rangle \hat{\mathbf{v}}$$
- This is a more difficult problem for an arbitrary vector representation $\mathbf{u} + \alpha \mathbf{v}$

9

9

The vector representation of a line

Point on the line closest to a given point

- For example, what is the point on the line

$$\begin{pmatrix} 7.8432 \\ -10.304 \\ -15.5424 \end{pmatrix} + \alpha \begin{pmatrix} 0.48 \\ -0.6 \\ 0.64 \end{pmatrix}$$
 that is closest to this point?

$$\mathbf{w} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$
 - Because $\mathbf{u}_\perp$ is orthogonal to $\hat{\mathbf{v}}$, we need only project $\mathbf{w}$ onto $\hat{\mathbf{v}}$:

$$\mathbf{u}_\perp + \langle \hat{\mathbf{v}}, \mathbf{w} \rangle \hat{\mathbf{v}} = \begin{pmatrix} 7.0928 \\ -11.616 \\ -16.2096 \end{pmatrix}$$

10

10

The vector representation of a line

Point on the line closest to a given point

- Given this closest point

$$\mathbf{u}_{\perp} + \langle \hat{\mathbf{v}}, \mathbf{w} \rangle \hat{\mathbf{v}} = \begin{pmatrix} 8.0928 \\ -10.616 \\ -15.2096 \end{pmatrix}$$

we also have that the coefficient of $\hat{\mathbf{v}}$ is

$$\alpha = \langle \hat{\mathbf{v}}, \mathbf{w} \rangle = 0.52$$
- Thus, if we are using meters,
 - this closest point is 0.52 m from the point $\mathbf{u}_{\perp}$
 - You can verify yourself that this is indeed the closest point to $\mathbf{w}$

11

11

The vector representation of a line

Minimum distance to a line from a given point

- You can also calculate the distance to the line as follows:

$$\sqrt{\|\mathbf{u}_{\perp}\|_2^2 + \|\mathbf{w}\|_2^2 - 2\langle \mathbf{u}, \mathbf{w} \rangle - \langle \hat{\mathbf{v}}, \mathbf{w} \rangle^2}$$
 - You can derive this yourself by calculating

$$\|\mathbf{u}_{\perp} + \langle \hat{\mathbf{v}}, \mathbf{w} \rangle \hat{\mathbf{v}} - \mathbf{w}\|_2^2 = \langle \mathbf{u}_{\perp} + \langle \hat{\mathbf{v}}, \mathbf{w} \rangle \hat{\mathbf{v}} - \mathbf{w}, \mathbf{u}_{\perp} + \langle \hat{\mathbf{v}}, \mathbf{w} \rangle \hat{\mathbf{v}} - \mathbf{w} \rangle$$
 - In this example, the distance was approximately 21.166 m

12

12



The vector representation of a line

$\mathbf{R}^n$

- Note that there is nothing in this topic specific to $\mathbf{R}^3$
 - Consequently, all these techniques can be used in $\mathbf{R}^n$

13 

13



The vector representation of a line

Summary

- Following this topic, you now
 - Understand the vector representation of a line
 - Know how to find the ideal representation
 - Can use this representation to find the point on the line closest to a given point $\mathbf{w}$
 - Are aware of a formula to calculate the minimum distance from $\mathbf{w}$ to that line

14 

14



The vector representation of a line

References

[1] [https://en.wikipedia.org/wiki/Line_\(geometry\)](https://en.wikipedia.org/wiki/Line_(geometry))

15 

15



The vector representation of a line

Acknowledgments

None so far.

16 

16

The vector representation of a line

Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas. Mathematical equations are prepared in MathType by Design Science, Inc. Examples may be formulated and checked using Maple by Maplesoft, Inc.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see <https://www.rbg.ca/> for more information.



17

17

The vector representation of a line

Disclaimer

These slides are provided for the NE 112 *Linear algebra for nanotechnology engineering* course taught at the University of Waterloo. The material in it reflects the authors' best judgment in light of the information available to them at the time of preparation. Any reliance on these course slides by any party for any other purpose are the responsibility of such parties. The authors accept no responsibility for damages, if any, suffered by any party as a result of decisions made or actions based on these course slides for any other purpose than that for which it was intended.

18

18